A brief overview of 2nd Order (or P-Delta) Analysis

by Richard Dobson - Senior Consultant, Trimble Solutions Corporation.

Kenny Arnott – Principal Structural Consultant, Trimble Solutions Corporation

(A version of this article was published in the Jan 2003 edition of New Steel Construction, at which time Richard Dobson was Technical Director at CSC (UK) Ltd)

Objective

The objective of this document is to provide a short introduction to engineers on the different methods of 2nd order analysis currently available in commercial software form. In order to provide a document which is able to be appreciated by a wide range of engineers, the document also includes a simple overview of what the P-delta effects are.

Introduction

30 years ago, engineers used simple hand calculations to determine design forces and moments, in many cases the word analysis was not mentioned even though it was being performed intrinsically.

Engineers today typically use linear elastic static (first order) analysis to determine design forces and moments resulting from loads acting on a structure.

First order analysis assumes small deflection behaviour; the resulting forces and moments take no account of the additional effect due to the deformation of the structure under load.

Second order analysis combines two effects to reach a solution:-

- large displacement theory; the resulting forces and moments take full account of the effects due to the deformed shape of both the structure and its members.
- “stress stiffening”; the effect of element axial loads on structure stiffness, tensile loads stiffening an element and compressive loads softening an element.

As structures become ever more slender and less resistant to deformation, the need to consider 2nd order, and to be more specific, P-delta effects arises. As a result, Codes of Practice are referring engineers more and more to the use of 2nd order analysis in order that P-delta and “stress stiffening” effects are accounted for when appropriate in design. This is as true in concrete and timber design as it is in the design of steelwork.

What are the P-delta effects ?

P-Delta is a non-linear (second order) effect that occurs in every structure where elements are subject to axial load. P-Delta is actually only one of many second-order effects. It is a genuine “effect” that is associated with the magnitude of the applied axial load (P) and a displacement (delta).

There are two P-Detal effects:-

- P-“BIG” delta effect (P-Δ) – a structure effect
- P-“little” delta (P-δ) – a member effect

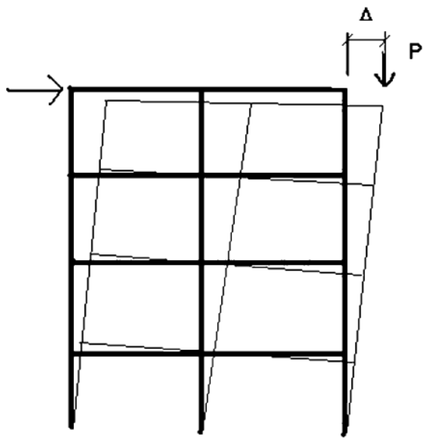
The magnitude of the P-delta effect is related to the:-

- magnitude of axial load P
- stiffness/slenderness of the structure as a whole.
- slenderness of individual elements

By controlling slenderness, the magnitude of the P-delta effect is often “managed” such that it can be considered negligible and then “ignored” in design; for instance, at the structure level by the use of more or heavier bracing, at the element level by increasing member size.

Most people understand P-Delta as -

- Frame deflects; Delta,
- Load P is then eccentric to the base - this introduces further moments or ‘second order effects’

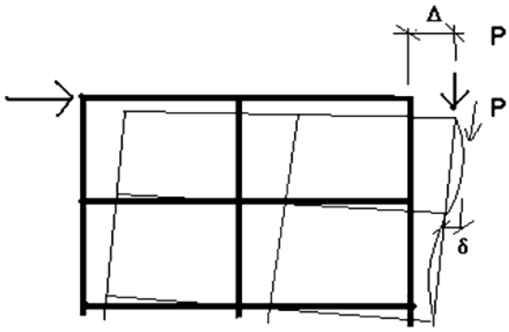


However, this only illustrates the P-“BIG” delta effect (P-Δ) . P-Δ is only part of the second order effect.

Consider again the “P-Delta” effect. Here we see the effects of both P-“BIG” delta (P-Δ) and P-“little” delta (P-δ).

What really happens is that the load deflection characteristic changes in the presence of axial load – the structure will deform more in the presence of axial load.

This effect occurs at both an overall structural level and an element level.



To obtain true design forces and moments, which accommodate all the P-Delta effects, then the analysis method used should account for both P-Δ and P-δ; both the deltas (Δ and δ) are inextricably linked – an increase in one brings about an increase in the other.

It is worth noting that where the P-Δ effect is managed in the structure to a level where it can be ignored then the P-δ is accounted for in the buckling curves used in design codes of practice. Importantly, the buckling curves also include the additional effect of initial member imperfection. Initial member imperfection is typically not included for in a second order analysis and thus it is important to design using the strut curves even if both P-Δ and P-δ are taken account of.

Do you need to use 2nd Order or P-Delta analysis?

Engineers have been aware of the P-Delta effects for many years. However, it is only relatively recently that the computational power to provide analytical approximations to this effect has become widely available.

In the absence of more rigorous analysis capabilities many design codes have in the past incorporated empirical checks and “Good Practice” design guidance to ensure that the magnitude of second order effects stay within limits that are inherently allowed for. Alternatively, they have made simple and approximate provision for them using methods like amplified sway or extended effective lengths.

While this has ensured safe design, it has perhaps muddied the waters when it comes to developing a clear understanding of the “effect”.

Methods of accounting for P-Delta effects.

It is an engineer’s judgement as to how accurately the second order effects need to be accounted for in determining design forces and moments.

The simplest way to make sure that one does not have to account for the second order effects is to ensure the structure and its elements are sufficiently stiff so that the effects can be considered as negligible. It is possible to spend time and effort avoiding the use of P-Delta analysis by proving that the effects will be negligible. But if the effects are negligible a P-Delta analysis will return very similar answers to a linear elastic analysis.

If they are not negligible, which design method should be used?

- Amplified sway
- Extended effective lengths
- 2nd Order Analysis

The first two of these are approximate and can be awkward to apply to a design unless design software tools handle the issues for you. For some structures these methods are deemed unsuitable and a second order analysis is required by the code of practice.

Is "P-Delta" Analysis the same thing in all analysis products ?

The simple answer is "NO" – the analysis procedures used to determine P-Delta effects vary from one piece of software to the next - watch out for limitations and conditions.

Several methods of allowing for second order effects in analysis have been developed. Some of these methods rely on a constrained problem or set of conditions, and will therefore have documented "limitations".

Remember that analysis models are by definition "modelling" of the real condition and hence always provide approximations to the real world.

None of the more approximate methods operate by recognising and adjusting the stiffness matrix on an element by element basis. To do so requires a more complex element matrix formulation such as the "Geometric (stress) Stiffness Matrix".

The P-Delta effect does not distinguish between types or directions of loading. It does not know about floors, floor levels, or the difference between a beam and a column. If the analysis makes assumptions or requires additional data input along these lines care should be used to understand and work within the limitations of the analysis.

Some P-Delta Methods

Four different analytical methods are considered below. The first two approximate the P-Delta effect using 1st order elastic analysis, thus care is needed in their use:-

- A "pseudo load" approach
- A "pseudo displacement" approach
- The two-cycle iterative method – accounting for geometric stres stiffness
- Non-linear static analysis – full Newton Raphson.

A "Pseudo-Load" Approach

Simple elastic analysis (small displacement) takes no account of this 'secondary' action on the frame. But, it is possible to begin to approximate the P-Δ effect in two ways using simple elastic analysis.

Typically this sort of approach relies on the structure being subject to predominantly gravity (vertical) loading.

Often it also relies on there being defined floor or diaphragm levels within the structure. An additional (Pseudo) horizontal load (F_h) at each level can be estimated as:

$$F_h = (P \times \delta) / h$$

P = The vertical load at that level (Floor).

d = The relative deflection (or drift) between that level and the level below.

h = The distance between levels (Floor to Floor height)

The calculation of d requires an initial analysis and then the structure is re-analysed with the added loads. The process can be repeated (further iteration) with further adjustment to the Pseudo Load.

In the end, the result could be accurate (if the structure fits in with the limitations), but there will be forces within the structure and base reactions that relate to entirely artificial loading.

REMEMBER this method only deals with the P-Δ effect and only on structures which are predominantly gravity loaded and which have clearly defined floors or diaphragm levels.

BS 5950 & Eurocode (UK Design Code) Note - the preceding method should not be confused with "Notional Loading" (BS 5950) or "Equivalent Horizontal Loads" (Eurocode) which are prescribed in the first instance to account for "practical imperfections".

A "Pseudo-Displacement" approach

Rather than introducing an artificial load to induce deformation, why not introduce artificial deflections?

Once again this would involve an initial elastic analysis to establish nodal deflections, the structural model is then re-built using this deflected geometry and is re-analysed elastically. This process can obviously be repeated progressively but since "stress stiffening" is not taken into account then solutions may not converge.

This approach will not give a good approximation to the P-Delta effect as it takes no account of the 'work done' to move the structure. At best it is a simple approximation. Inaccuracies increase as "second order" effects increase.

REMEMBER this method is only approximate since "stress stiffening" is not taken into account and it only deals with the P-Δ effect.

The Two Cycle Iterative Method

P-Delta is the only non-linear effect addressed by this analysis type. It does however account for both the P-Δ and P-δ effects.

The Two-Cycle Iterative Method (Chen and Lui (1991)) requires a two-pass analysis procedure. In the first phase, the system equilibrium equations for the linear static analysis are solved for the nodal displacements for all of the user-selected load cases and load combinations.

Once the nodal displacements are obtained, the element membrane forces are calculated and used to form the element geometric stiffness matrices. The element geometric (stress) stiffness matrix includes both the P-Δ and P-δ effects as well as accounting for "stress stiffness".

When a general "Geometric (stress) Stiffness Matrix" is used in the method, there are no significant limitations on it's use or applicability, unless gross deformation occurs when a full non-linear iterative solution is more appropriate.

Non-Linear Static (Full Newton Raphson) Analysis

A Full Non Linear Iterative Solution allowing all sorts of other non-linear conditions to be accounted for simultaneously, in addition to both the P-Δ and P-δ effects.

The system equilibrium equations for the non-linear static analysis are

$$\{F\}_s - \{R\}_s = 0$$

where the system load vector $\{R\}_s$ includes the contributions of all externally applied nodal forces:

1. Applied nodal loads
2. Loading due to static acceleration fields (such as gravity)
3. Element thermal/pressure loads

…and vector $\{F\}_s$ includes the nodal forces that correspond to the element stresses in this configuration. This vector is known as the internal load vector.

The solution of the non-linear equation is carried out in an incremental step-by-step analysis. Additionally the total applied loads can be divided into a number of load steps (n). The most popular method of solution for non-linear equations is the Newton Raphson method.

When a general "Geometric (stress) Stiffness Matrix" is used in the method, there are no significant limitations on its use or applicability.

Additional Note

All the above analysis methods assume a linear material behaviour. Elastic-plastic or elastoplastic material behaviour requires additional consideration. For portal frames, this is dealt with fully in SCI publication P292 "In plane stability of Portal Frames to BS5950-1:2000"